Modele pentru rezolvarea problemelor şi redactarea soluţiilor

1. Calculaţi:

a) $\lim\limits_{x\to 0}\dfrac{e^x-1}{3\sin x}$;   b) $\lim\limits_{x\to 0}\dfrac{2^x+3^x-2}{x^2+3x}$;

c) $\lim\limits_{x\to 0}\dfrac{x^2(\sin x-e^x)}{\sin x+x^2e^x}$;   d) $\lim\limits_{x\to 0}\dfrac{x^2-x^2\cos x}{x^2\sin^2 x+\sin^3 x}$;

e) $\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}-\ln\cos x}{x^2}$.

Soluţie:

a) $$\lim_{x\to 0}\frac{e^x-1}{3\sin x}=\lim_{x\to 0}\frac{e^x-1}{x}\cdot\frac{1}{3}\lim_{x\to 0}\frac{x}{\sin x}=1\cdot\frac{1}{3}\cdot 1=\frac{1}{3};$$

b) $$\lim_{x\to 0}\frac{2^x-1+3^x-1}{x^2+3x}=\lim_{x\to 0}\frac{2^x-1}{x}\cdot\frac{x}{x^2+3x}+\lim_{x\to 0}\frac{3^x-1}{x}\cdot\frac{x}{x^2+3x}=$$

$$=\lim_{x\to 0}\frac{2^x-1}{x}\cdot\lim_{x\to 0}\frac{1}{x+3}+\lim_{x\to 0}\frac{3^x-1}{x}\cdot\lim_{x\to 0}\frac{1}{x+3}=\ln 2\cdot\frac{1}{3}+\ln 3\cdot\frac{1}{3}=$$

$$=\frac{1}{3}\ln 6=\ln\sqrt[3]{6};$$

c) $$\lim_{x\to 0}\frac{x^2\left(\sin x-e^x\right)}{\sin x+x^2e^x}=\lim_{x\to 0}\frac{x\left(\sin x-e^x\right)}{\frac{\sin x}{x}+xe^x}=\frac{0}{1+0}=0;$$

d) $$\lim_{x\to 0}\frac{x^2(1-\cos x)}{x^2\sin^2 x\left(1+\frac{\sin x}{x}\cdot\frac{1}{x}\right)}=\lim_{x\to 0}\frac{1-\cos x}{(1-\cos x)(1+\cos x)\left(1+\frac{\sin x}{x}\cdot\frac{1}{x}\right)}=$$

$$=\lim_{x\to 0}\left(\frac{x}{1+\cos x}\cdot\frac{1}{x+\frac{\sin x}{x}}\right)=0\cdot\frac{1}{0+1}=0;$$

e) $$\lim_{x\to 0}\left(2\cdot\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}\cdot 4}-\frac{\ln(1+(\cos x-1))}{\cos x-1}\cdot\frac{(\cos x-1)}{x^2}\right)=$$

$$=\lim_{x\to 0}\frac{1}{2}\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2-\lim_{x\to 0}\frac{\ln(1+(\cos x-1))}{\cos x-1}\cdot\frac{-2\sin^2\frac{x}{2}}{\frac{x^2}{4}\cdot 4}=$$

$$=\frac{1}{2}-1\cdot\lim_{x\to 0}\left(-\frac{1}{2}\right)\cdot\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}=\frac{1}{2}+\frac{1}{2}\cdot 1=1.$$

2. Calculați:

a) $\lim_{x\to\infty}\frac{\ln^2 x^2}{x^n}, n\in\mathbb{N}^*$;

b) $\lim_{x\to\infty}\frac{e^x}{x^{n-1}}\sin\frac{1}{x}, n\in\mathbb{N}^*$;

c) $\lim_{x\to\infty}\frac{x^3+2x^2+x-2}{e^x+x+2}$;

d) $\lim_{x\to\infty}\frac{\ln(x^2+x+2)}{\ln(x^2-x+2)}$.

Soluție:

a) $$\lim_{x\to\infty}\frac{(\ln x^2)^2}{x^n}=\lim_{x\to\infty}\frac{(2\ln x)^2}{x^n}=\lim_{x\to\infty}4\left(\frac{\ln x}{x^{\frac{n}{2}}}\right)^2=4\cdot 0=0;$$

b) $$\lim_{x\to\infty}\frac{e^x}{x^n}\cdot x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{e^x}{x^n}\cdot\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\infty\cdot 1=\infty;$$

c) $\lim\limits_{x\to\infty} \dfrac{e^x\left(\dfrac{x^3}{e^x}+2\dfrac{x^2}{e^x}+\dfrac{x}{e^x}-\dfrac{2}{e^x}\right)}{e^x\left(1+\dfrac{x}{e^x}+\dfrac{2}{e^x}\right)} = \lim\limits_{x\to\infty} \dfrac{\dfrac{x^3}{e^x}+\dfrac{2x^2}{e^x}+\dfrac{x}{e^x}-\dfrac{2}{e^x}}{1+\dfrac{x}{e^x}+\dfrac{2}{e^x}} = \dfrac{0}{1} = 0;$

d) $\lim\limits_{x\to\infty} \dfrac{\ln x^2\left(1+\dfrac{1}{x}+\dfrac{2}{x^2}\right)}{\ln x^2\left(1-\dfrac{1}{x}+\dfrac{2}{x^2}\right)} = \lim\limits_{x\to\infty} \dfrac{\ln x^2+\ln\left(1+\dfrac{1}{x}+\dfrac{2}{x^2}\right)}{\ln x^2+\ln\left(1-\dfrac{1}{x}+\dfrac{2}{x^2}\right)} =$

$$= \lim_{x\to\infty} \frac{\ln x\left[2+\dfrac{\ln\left(1+\dfrac{1}{x}+\dfrac{2}{x^2}\right)}{\ln x}\right]}{\ln x\left[2+\dfrac{\ln\left(1-\dfrac{1}{x}+\dfrac{2}{x^2}\right)}{\ln x}\right]} = \lim_{x\to\infty} \frac{2+\dfrac{\ln\left(1+\dfrac{1}{x}+\dfrac{2}{x^2}\right)}{\ln x}}{2+\dfrac{\ln\left(1-\dfrac{1}{x}+\dfrac{2}{x^2}\right)}{\ln x}} = \frac{2+0}{2+0} = 1.$$

3. Determinați $a \in \mathbb{R}$, astfel încât:

a) $\lim\limits_{x\to 0}\dfrac{1-\cos ax}{x^2} = 2;$

b) $\lim\limits_{x\to 1} f(x) = 1$, unde $f(x) = \begin{cases} 3-ax, & x<1 \\ \sqrt{x^2-2ax+a^2}, & x\geq 1 \end{cases}.$

Soluție:

a) $\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{ax}{2}}{\dfrac{a^2x^2}{4}}\cdot\dfrac{a^2}{4} = 2$. Rezultă că $2\cdot\dfrac{a^2}{4} = 2 \Rightarrow a^2 = 4 \Rightarrow a \in \{-2, 2\};$

b) $\lim\limits_{\substack{x\to 1\\ x<1}} (3-ax) = 3-a$ și $\lim\limits_{\substack{x\to 1\\ x<1}}\sqrt{x^2-2ax+a^2} = \sqrt{1^2-2a+a^2} =$

$=\sqrt{(a-1)^2} = |a-1|$. O funcție are limită într-un punct $x_0 = 1$ dacă $l_s = l_d$.

Problema cere ca $l_s = l_d = 1 \Rightarrow 3 - a = 1 \Rightarrow a = 2$ și $|a-1| = 1 \Rightarrow a = 2$ sau $a = 0$. Rezultă că $a = 2$.

4. Pentru orice $n \in \mathbb{N}$, $n \geq 2$, calculați:

a) $\lim\limits_{x \to \infty} \dfrac{\left(3x+\sqrt{x^2-2}\right)^n+\left(3x-\sqrt{x^2-2}\right)^n}{x^n}$;

b) $\lim\limits_{x \to 0} \dfrac{1-\sqrt{1+x}\cdot\sqrt[3]{1+x}\cdot\sqrt[4]{1+x}\cdot\ldots\cdot\sqrt[n]{1+x}}{x}$.

Soluție:

a) $\lim\limits_{x \to \infty} \dfrac{\left(3x+x\sqrt{1-\frac{1}{x^2}}\right)^n+\left(3x-x\sqrt{1-\frac{1}{x^2}}\right)^n}{x^n} =$

$= \lim\limits_{x \to \infty} \dfrac{x^n\left[\left(3+\sqrt{1-\frac{1}{x^2}}\right)^n+\left(3-\sqrt{1-\frac{1}{x^2}}\right)^n\right]}{x^n} =$

$= \lim\limits_{x \to \infty}\left(3+\sqrt{1-\frac{1}{x^2}}\right)^n+\left(3-\sqrt{1-\frac{1}{x^2}}\right)^n = 4^n + 2^n = 2^n(2^n+1).$

b) $\lim\limits_{x \to 0} \dfrac{1-(1+x)^{\frac{1}{2}}\cdot(1+x)^{\frac{1}{3}}\cdot(1+x)^{\frac{1}{4}}\cdot\ldots\cdot(1+x)^{\frac{1}{n}}}{x} = \lim\limits_{x \to 0} - \dfrac{(1+x)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}}-1}{x} =$

$= -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}\right).$